Analysis of Linear Facial Growth

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Many investigators have described changes in size of component parts of the face which occur as a consequence of growing up. It has been the opinion generally that increase in these linear dimensions does not occur at a uniform rate but proceeds with alternate acceleration and deceleration.[3,4,5,7,10,12] However, Brodie[1] considered that the rate of growth diminished progressively at least up to eight years of age. Most studies have not described the rate of change in quantitative terms or have used methods which lack a firm theoretical base. Our objective is to describe in a concise, mathematical way growth rates of a number of facial parts over a relatively long period of time. It may be possible in this way to determine the exact nature of the linear changes occurring in the individual face during its developmental history and to describe the range of experience encountered in a representative sample.

The ideas developed by Samuel Brody[2] in *Bioenergetics and Growth* seem to offer a fresh approach to the analysis of growth in individuals. All of the theory described below derives from chapter sixteen of his book where it is presented in detail and with great clarity. Some of the more general concepts are mentioned first to provide a proper background for the detailed analysis which follows.

The age curve of growth has an S-like shape and may be divided into two principal segments, the first of increasing slope (designated as the self-accelerating phase of growth) and the second of decreasing slope (designated as the self-inhibiting phase. The general slope of the age curve may thus be said to be determined by two opposing forces. The first portion manifests itself in the tendency of the reproducing units to reproduce at a constant percentage rate indefinitely when permitted to do so. In the absence of inhibiting forces the number of new individuals produced per unit time is always proportional to the number of reproducing units. In other words, the percentage growth rate tends to remain constant. But there comes a time, marked by the inflection in the curve, when the increase tends to be proportional to the available resources necessary for growth. So it is the environment which exerts the limiting influence after the inflection in the curve.

The inflection in the age curve represents the position at which the increase in growth velocity ceases and the decrease in velocity has not yet begun, i.e., acceleration is zero. The inflection represents the position at which gains are most rapid and perhaps most economical. At least one physiological stage through which mammals pass at this time is puberty which in human children occurs between twelve and fifteen years of age.

Previous analyses of growth rates have been handicapped by expressing change during finite periods of time rather than in infinitesimal units. Calculations have been made, sometimes with refinements, along the lines described by Minot.[8] The notations used below refer to *weight* since the concepts were first described in relation to body weight.

Average *absolute* growth rate =

$$\frac{W_2 - W_1}{t_2 - t_1}$$

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Where W_1 is weight at initial time (t_1) and W_2 is weight at some later time (t_2). But the idea of *average* rate is an abstraction and, when the average extends over a considerable period of time, it gives no idea of the actual rate at any time within the period. The shorter the time interval for which the average is calculated, the more nearly it approaches the true value. When reduced to an interval, dt, so short that there is no time for the velocity of growth to change, the true growth rate is then *instantaneous growth rate,* dW/dt.

Relative growth rate is usually represented by the following expression:

$$R = \frac{W_2 - W_1}{W_1}$$

Where R = Average relative growth rate or average percentage growth rate when multiplied by 100.

R is represented by the weight gain during a given time interval divided by the weight of the organism at the beginning of the time interval. Consequently, this is not a true or instantaneous growth rate but the conventional growth rate. The conventional and true percentage growth rates are nearly identical when the weight gain, $W_2 - W_1$, is very small in comparison with the weight of the organism. But when the weight gain is relatively large compared with body weight, the conventional growth rate computed from this equation will be exaggerated, because the weight at the beginning of the time interval, W_1, existed a relatively long time before the final observation. Another fault in this calculation is its failure to recognize that the physiological significance of a physical unit of time changes rapidly with age. The growth rates computed by the above equation represent continuously decreasing physiological time intervals although they employ constant physical time intervals. The use of instantaneous rates eliminates this discrepancy between physiological and physical time.

If in place of finite weight gain,

$$\frac{W_2 - W_1}{t_2 - t_1}$$

we use instantaneous weight gain, dW/dt, and divide the instantaneous gain by the weight at the time of the gain, the instantaneous relative growth rate, k, results. When multiplied by 100, k is called the instantaneous percentage rate of growth.

$$k = \frac{dW/dt}{W} = \text{instantaneous relative growth rate} \qquad (1)$$

$$\frac{dW}{dt} = k\,W \quad \text{or} \quad \frac{dW}{W} = k\,dt \qquad (2)$$

$$\int_A^W \frac{dW}{W} = k \int_o^t dt \qquad (3)$$

The infinite number of infinitesimal instantaneous rates are added or integrated resulting in the following:

$$\ln W = \ln A + kt \qquad (4)$$

$$W = A + e^{kt} \qquad (5)$$

In equation 4

ln W = natural logarithm of weight (W) at time (t)

ln A = natural logarithm of W when t = o

e = base of natural logarithms

$$k = \frac{\ln W_2 - \ln W_1}{t_2 - t_1} \qquad (5a)$$

Thus, the practically impossible task of measuring the instantaneous growth rate is made possible by a mathematical device.

Equation 2 and its integrated form 5 are essentially the equations of the physical chemist for the kinetics of monomolecular change. The principle of mass action is applicable to the multiplication rate of any category of reproducing units whenever the reproduction rate tends to be directly proportional to the number of reproducing units.

Equation 5 may be fitted to the data by the method of least squares or graphically. In either case it is first written in linear form by taking logarithms of both sides:

$$\ln W = \ln A + kt \qquad (4)$$

which has the same form as the linear equation $y = a + bx$ where ln W corresponds to y, ln A to a and kt to bx. If the logarithms of W are plotted against the corresponding values of t, a straight line results of slope k and intercept ln A and the equation is fitted graphically to the data. Wherever the equation fails to represent the data, there is deviation from a straight line. The work of looking up logarithms can be saved by plotting the points on semilogarithmic paper bearing in mind that common logarithms can be converted into natural logarithms by multiplying by a constant factor, 2.303.

In deriving an expression for the self-inhibiting phase, it is reasonable to assume that the instantaneous growth velocity, dW/dt, will be proportional to the limitations of the environment such as food supply, living space, etc. The arbitrary constant, A, may be used to represent the mature weight of the animal under a given set of conditions; W can represent the weight of the animal at a given time; and (A-W) then represents the amount of growth yet to be realized for attainment of the mature weight.

$$\frac{dW}{dt} = -k\,(A-W) \qquad (6)$$

$$-k = \frac{dW/dt}{A-W} \quad \text{is integrated}$$

$$\ln (A-W) = -kt + \ln B \qquad (7)$$

Where B is an integration constant

$$A-W = Be^{-kt}$$

$$W = A - Be^{-kt} \qquad (8)$$

The fit of equation 8 to growth data may be examined by the graphic method as explained before. (A-W) is plotted against age, t, on arithlog paper. A straight line results if the proper value of A is chosen, and if the equation represents the data. Several values of A are chosen and A-W plotted. There is an upward curvature for high values of A, a downward curvature for low values of A, and a linear distribution for the correct value of A.

The theoretical concepts described above were developed to describe and analyze weight changes in the individual. There is sufficient justification to use these ideas in the study of linear head changes. This question will be discussed later. Data collected on one person over a period of eighteen years were subjected to analysis to demonstrate the method.

Method

Eleven linear measurements were obtained from tracings of eighteen consecutive lateral cephalometric roentgenograms of a white male. The series began at three months of age, was followed by two records at three month intervals, one nine months later, one after eighteen months, and thereafter at yearly intervals for the most part, until the subject was eighteen years, four months old.

The anthropologic landmarks (Fig. 1) were those customarily used: nasion, sella, basion, anterior nasal spine, point A (subspinale), prosthion, infradentale, mandibular symphyseal

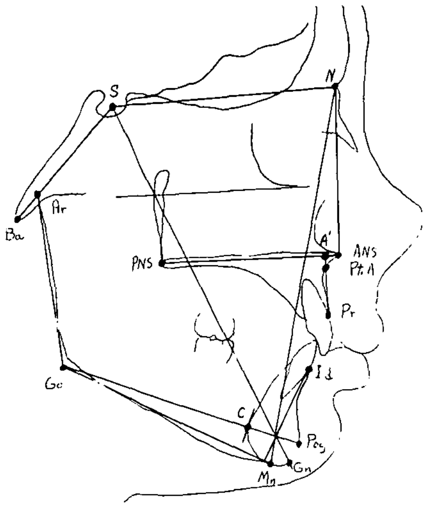

Fig. 1 The anthropologic landmarks and constructed points used in the study.

outline, pterygomaxillary fissure, pogonion, menton, gnathion, posterior nasal spine, articularé, [for reasons of expedience, the point was taken as the intersection between the Ba-S line and the tangent to the posterior margins of the ascending rami] and gonion. Two additional points were C, the point on the Go-Pog line at which a perpendicular is tangent to the posterior contour of the symphyseal outline and A′, the intersection on the ANS-PNS line of a perpendicular drawn from point A.

In most cases each linear measurement could be considered to represent a specific anatomic part. For example, N-S represented the anterior cranial base; Id-Mand. plane signified the height of the anterior mandibular body; and Ar′-Go represented the ramal height. A′-PNS was used as a more accurate representation of the length of the body of the maxilla than ANS-PNS. Go-C was employed in order to remove, insofar as possible, the anteroposterior component of lateral symphyseal growth. S-GN was alone in lacking a more or less discrete anatomic counterpart.

All measurements were reduced to their unmagnified values in the customary manner using the formula:

$$X' \cdot \frac{a}{a+b} = X$$

Where X′ = observed length

a = anode to median sagittal plane distance

b = ML distance

X = corrected length.

These figures are listed in Table I with age expressed in decimal values according to the method suggested by Thurow.[11]

Findings

The data were plotted on arithmetic rectangular coordinates and smooth curves were drawn by inspection (Figs. 2 and 3). The regular disposition of the points along most of the curves was noteworthy and along one, the N-S length, the smoothness of distribution approached that of some physical quantity. Most divergence occurred in the Ba-S data which is probably a reflection of the inherent difficulty in locating basion. There was more than usual scatter around Ar′-Go which may have been caused by employing Ar′ rather than articulare as customarily defined. Finally, scatter and breaks in the curves involving infradentale and prosthion are easily explained in terms of alveolar bone changes at the time of deciduous and permanent incisal eruption.

All of the curves were parabolic in nature and represented that portion of the S-shaped curve after the point of inflection which, therefore, must have occurred prenatally. In only two dimensions, N-Mn and N-ANS, did a straight line define the curve for an extended period of time. Both of these showed a long linear segment extending from about four to thirteen years of age. The shape of the distance-travelled curves varied from a resemblance to total

TABLE I

Corrected Measurements in MM, Case B-2817

Age, Yrs.	N-S	Ba-S	N-Mn	N-ANS	A-PNS	Pr-Pal Pl.	Id-MN Pl.	Go-Pog	Go-C	Ar-Go	S-Gn
0.250	47.4	23.8	65.6	28.5	33.8	12.4	19.3	35.4	24.2	23.2	64.1
0.475	50.4	27.5	72.1	32.4	34.6	13.0	20.3	37.4	26.4	25.3	70.2
0.746	55.0	29.1	79.6	34.6	34.8	9.3	18.1	39.2	28.2	27.8	77.5
1.469	57.7	31.3	80.5	36.3	38.0	12.9	19.1	43.4	31.4	29.6	82.6
3.036	62.0	32.4	83.6	38.9	38.9	12.9	22.2	50.9	36.7	32.9	90.0
3.560	63.2	31.4	86.6	40.9	40.7	13.4	22.5	52.8	38.5	35.2	92.9
4.003	63.6	30.9	86.4	40.0	40.0	13.1	22.2	54.7	40.6	35.9	93.5
4.997	65.5	36.6	90.1	43.1	42.0	13.8	23.2	54.7	42.7	33.8	97.3
6.995	67.9	36.9	93.7	45.1	45.1	13.8	21.6	62.1	47.3	36.9	101.9
8.016	69.4	38.9	96.8	46.6	45.5	11.6	23.9	65.2	49.4	37.3	106.2
9.003	70.4	40.3	99.3	48.4	47.4	11.6	23.0	65.5	50.3	39.9	109.0
10.082	70.6	37.3	101.3	48.8	48.3	12.9	23.3	67.6	52.6	41.1	111.0
11.113	71.2	37.5	103.4	51.3	50.1	12.5	23.9	69.3	53.7	42.3	113.3
11.995	72.0	41.4	105.3	52.1	49.5	13.6	24.9	71.3	55.5	41.2	114.5
13.130	72.6	39.0	108.3	54.8	50.9	13.6	27.1	74.3	58.0	44.3	118.2
15.008	74.1	43.2	116.2	56.8	51.8	14.5	28.1	76.4	59.8	50.7	127.2
16.102	75.2	44.4	114.4	56.6	51.8	15.1	27.5	78.3	61.7	48.1	126.0
18.377	74.6	46.4	116.2	58.1	52.1	14.6	28.0	78.8	61.8	50.9	126.9

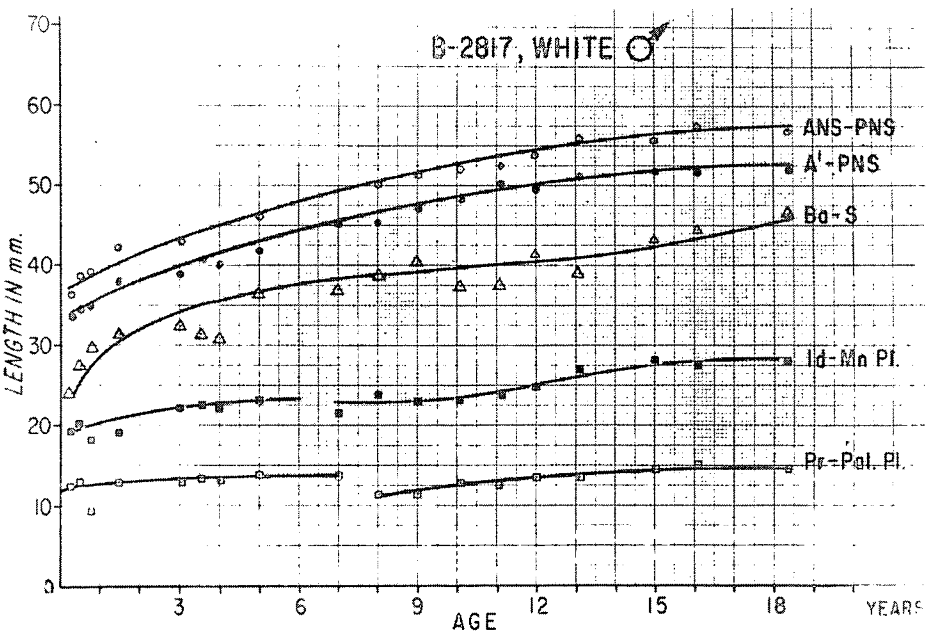

Fig. 2 Distance-travelled curves for several corrected linear measurements obtained by inspection.

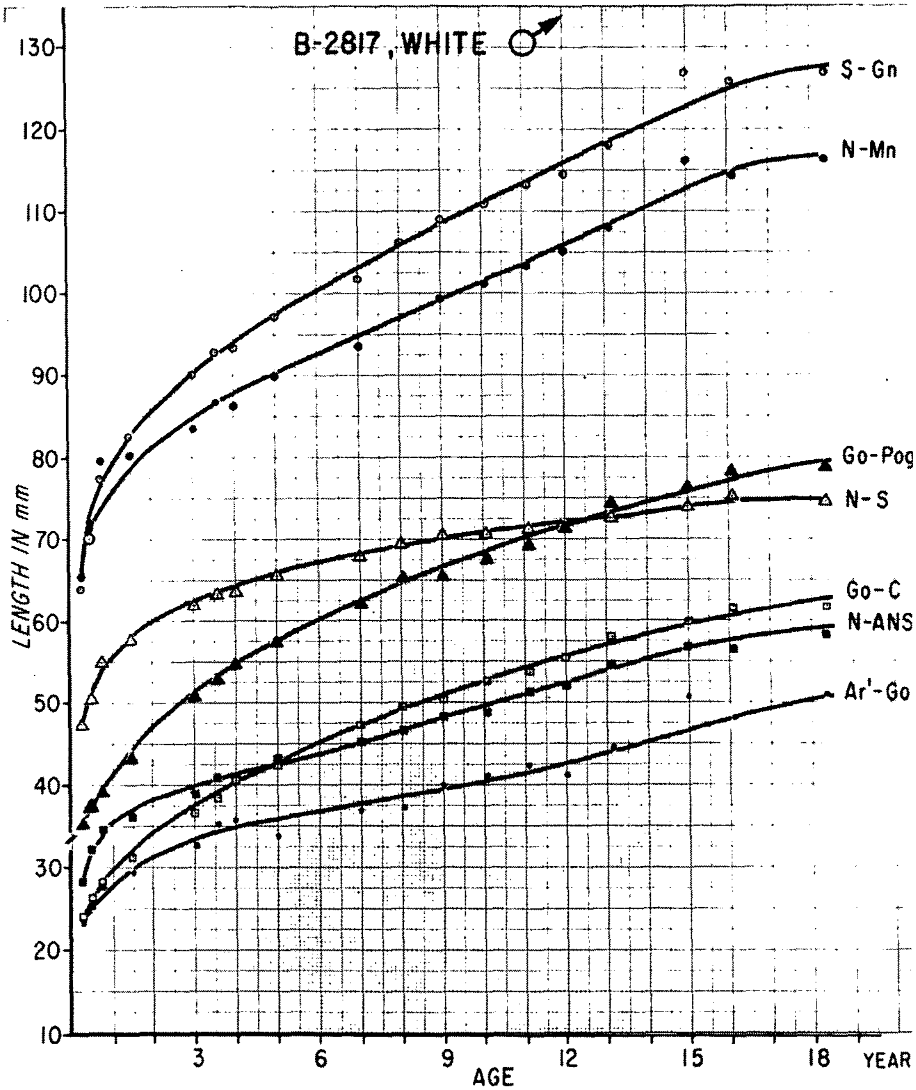

Fig. 3 Distance-travelled curves for several corrected linear measurements obtained by inspection.

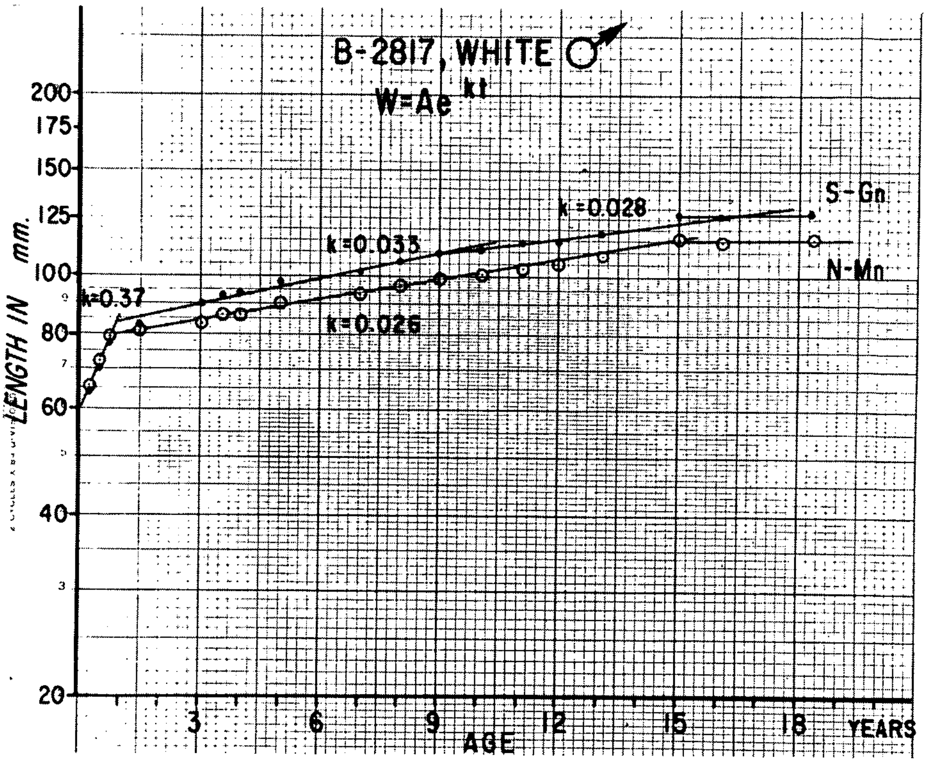

Fig. 4 Corrected linear dimensions plotted on semilogarithmic paper to the equation ln W = ln A + kt and curves drawn by inspection. The instantaneous relative growth rate, k, was calculated for each linear section. Note the progressive reduction in rate with increase in age.

body height, sitting height, arm length curves[2] in the case of S-Gn and N-Mn dimensions, to a similarity with the neural curve of Scammon for the N-S dimension. The other curves had characters of their own which could not be catalogued according to any one of the classical types of tissue growth curves.

Semilogarithmic coordinates were used in an attempt to determine the value of k in equation, $\ln W = \ln A + kt$. So much scatter was encountered in the Ba-S data that no attempt to plot a curve was made. The graphs shown in Figures 4, 5, and 6 indicate that most curves could be divided into several linear sections. This meant that the instantaneous relative growth rate was constant in each section and could be calculated according to equation 5a. The values for k are indicated in the respective figures.

Transition from one growth rate to another was abrupt and occurred at about one to one and one-half years of age in eight of the dimensions. Six curves showed a change in k at the fourteen-fifteen year age level. In all cases the k values diminished progressively with increasing age. There was no indication of a pubertal spurt in any curve but this end of the curve was less trustworthy because it lay in the region better analyzed by the self-inhibitory formula. In addition, the number and location of observations here were not as complete as might have been desired.

The data were replotted using the

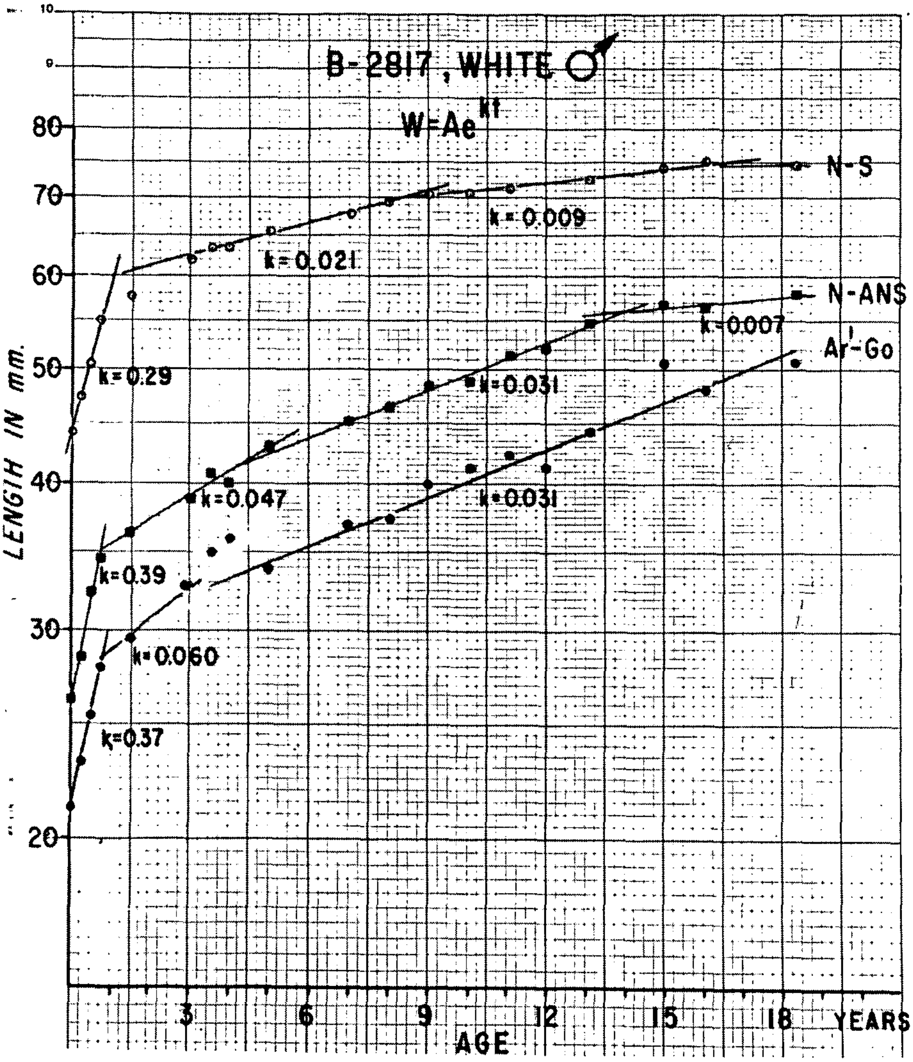
B-2817, WHITE ♂
W=Ae^kt
N-S
N-ANS
Ar'-Go
k=0.29
k=0.021
k=0.009
k=0.007
k=0.031
k=0.047
k=0.031
k=0.39
k=0.060
k=0.37
LENGTH IN mm.
AGE
YEARS
80
70
60
50
40
30
20
3
6
9
12
15
18

Fig. 5

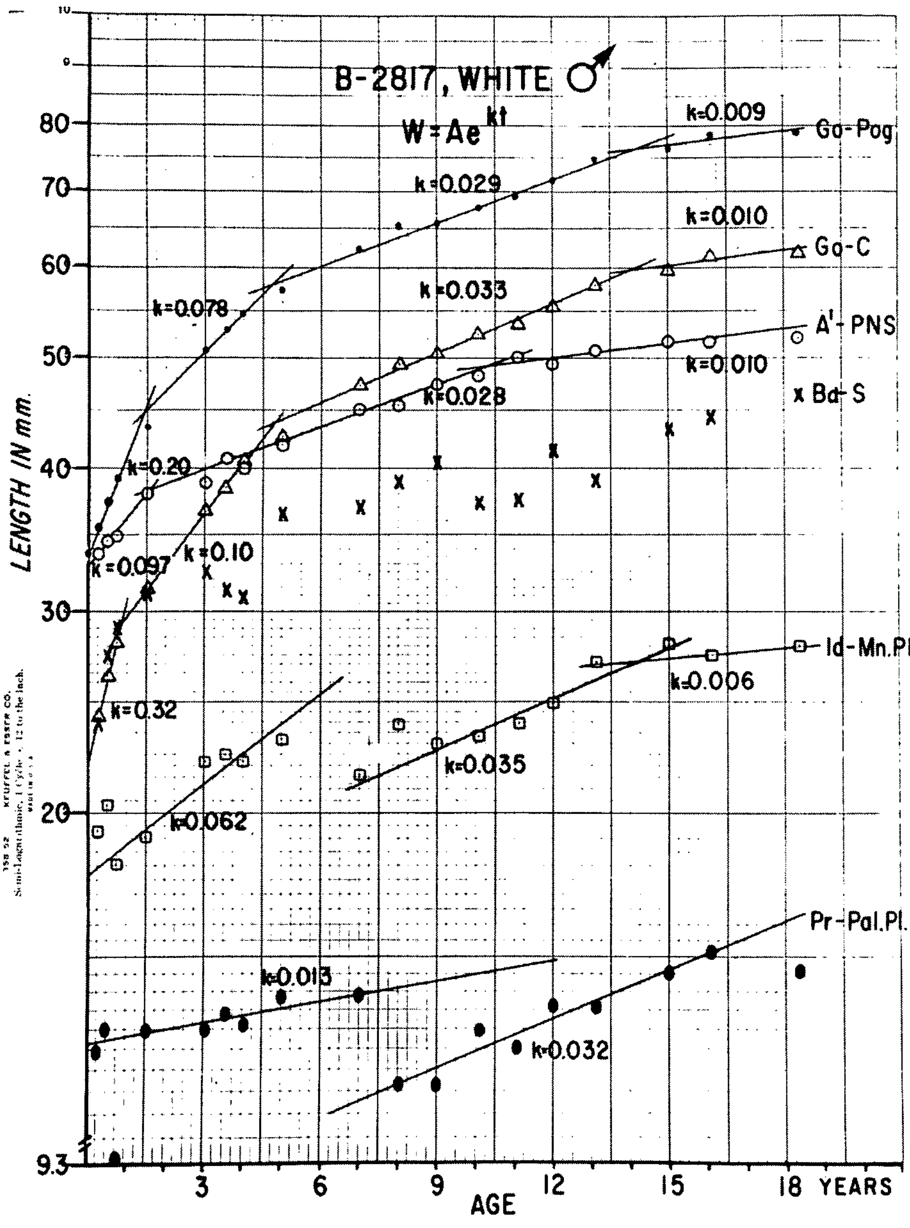
B-2817, WHITE ♂
W = Ae^kt
k=0.009
Go-Pog
k=0.029
k=0.010
Go-C
k=0.033
k=0.078
A'-PNS
k=0.010
k=0.028
x Bd-S
k=0.20
k=0.097
k=0.10
k=0.32
Id-Mn.Pl
k=0.006
k=0.035
k=0.062
Pr-Pal.Pl.
k=0.013
k=0.032
LENGTH IN mm.
80
70
60
50
40
30
20
9.3
3
6
9
12
15
18
YEARS
AGE

Fig. 6

self-inhibitory phase expression (equation 7) to examine the latter portion of the curves with more theoretical validity. Figure 7 shows the plot of N-S substituting 75.2 (the maximum recorded length), 80, and 100 successively for A. The first line curves downward indicating that the value of A is too small. The subject undoubtedly is not completely matured but an addition of 4.8 mm (resulting in A = 80) might be a maximum expected value. The curve is corrected somewhat but still curves downward at its extremity indicating an increase in the constant and hence an increased instantaneous relative growth rate. If this is evidence of a pubertal spurt, as it seems to be, a greater saturation of observations would be required during the critical period to document the change adequately. Only the uppermost curve indicates the magnitude of adjustment when A = 100 is chosen.

Figure 8 contains similar plots of the other dimensions where A was taken as the next whole number above the largest recorded observation in each series. Several curves have been drawn and illustrate the general tendency to behave as N-S. The same considerations probably are working here to influence linearity of the plot and hence behavior of the constant, k.

Discussion

The face at birth is a portion of a more or less spherical body, the head. While changes in size and in proportion do occur in the transition from the infant to the mature adult, the over-all changes are small in relation to change in the rest of the body. It is not surprising that facial dimensions at birth should already be beyond the inflection point of the sigmoid growth curve. The face is a complex association of many tissues and parts so that deviation from the classically described type of somatic growth curves could well be expected. If the results in one case can be any judge, the regularity of the process of linear increase is impressive.

The crucial point as far as our present purposes are concerned is whether the equations developed for the self-accelerating phase of body growth can be applied with validity to the face in that period from birth to the pubertal spurt. Unfortunately, there is no pat answer. I am inclined to give a positive one. It seems illogical to consider the face as a young adult and the rest of the newborn as an infant.

Application of the principle of mass action and the two phases of growth has been notably successful in describing changes in weight, numbers, and chemical constitution of a wide variety of living forms. This includes bacteria, human populations, a number of animal species, e.g., chick, rat, rabbit, cattle, horse, etc., and often includes consideration of prenatal as well as postnatal development stages. It is tempting to extend such a utilitarian approach to problems of facial change.

Many difficulties are encountered in determining what biological assumptions can be made about growth metabolism.[6] Multiplicative increase of cells has been assumed but, in fact, certain cells, e.g., muscle cells, do not multiply after fetal life but enlarge by incorporating and absorbing material. The skeleton which constitutes a large part of the body consists of material which is inert in a multiplicative sense. This is necessarily so since the accretionary nature of osteogenesis excludes an interstitial mode of formation. However, bone can be viewed as the resultant of the self-multiplicative processes of the surrounding osteogenic tissues. In this case, the growth of the bone is not measured itself but the relatively permanent record of the osteogenic tissue growth.[9]

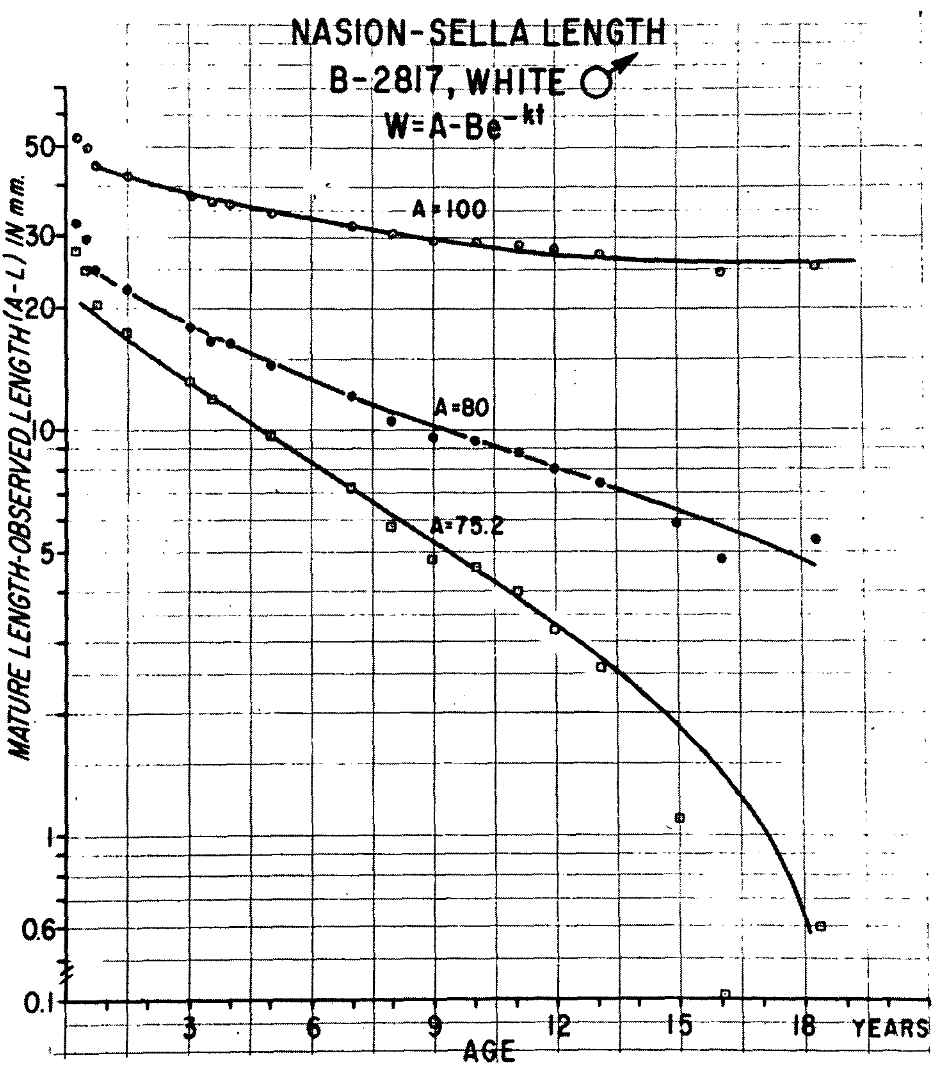

Fig. 7 Corrected linear dimensions for N-S plotted using the expression for the self-inhibitory phase, ln (A-W) = -kt + ln B. A represents the value for the mature length and is taken successively as the largest observed value for N-S (75.2), the largest expected value (80), and a value beyond reasonable expectation (100). Note that when A=80 the curve tends to be more linear but has some suggestion of a late growth spurt in the region of 16 years.

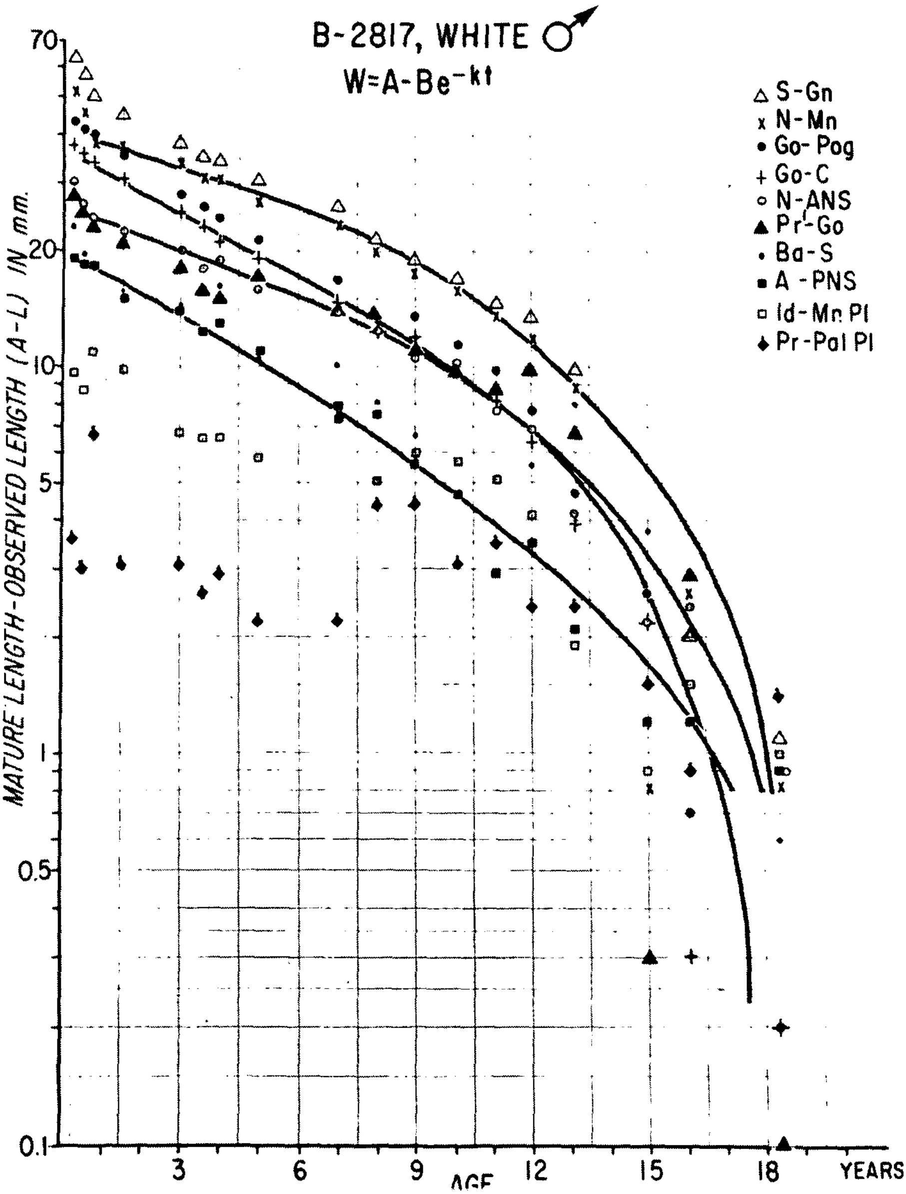

Fig. 8 Corrected linear dimensions plotted according to the expression for the self-inhibitory phase, ln (A-W) = -kt + ln B. In every case A was taken as the next largest whole number above the highest value observed. The downward course of the curves may be the result of selecting too small values for the A constants but it may also be a reflection of some increase in the growth rates during the later years.

If the theoretical base is sound, then we must agree with those who postulate a decreasing facial growth gradient but with the qualification that it occurs stepwise rather than continuously. Analysis of the self-inhibitory phase of facial growth throws no light on the rates occurring at puberty because of lack of sufficient data.

If a less general approach is indicated, the following observations have some pertinence. Description of the entire curve beginning at conception has never been achieved by a single equation. The curve has always had to be divided into segments for each of which equations have had to be devised. The object is to secure a curve which provides (1) a good fit, (2) has a reasonably simple functional expression, and (3) has relatively few constants whose meanings are clear and significant biologically. A well-chosen curve could have one parameter characterizing the time of onset of the adolescent spurt, another the rate of growth through the cycle, another its peak intensity, etc. These problems are dealt with by Israelsohn[6] in some detail.

Summary and Conclusions

The shape of the distance-travelled curve for eleven facial dimensions observed in one subject over a period of eighteen years varied from a resemblance to certain body length curves in boys (S-Gn and N-Mn) to a similarity with the neural curve of Scammon (N-S). The others had a character of their own which could not be catalogued according to any one of the classical types of tissue growth curves. In every instance except one, the points were distributed regularly along the constructed curves.

The curves were parabolic in character and seemed to represent the portion of the S-shaped growth curve which occurred after the major inflection point. The self-accelerating ($W = Ae^{kt}$) and self-inhibitory ($W = A\text{-}Be^{-kt}$) phase formulae were developed to describe weight change, but it seemed reasonable to apply them to linear facial changes.

The instantaneous relative growth rates of the dimensions studied were constant over certain sections of the curves, changed abruptly and decreased progressively with age. We would agree with those who postulate a decreasing facial growth gradient but with the qualification that it occurs stepwise rather than continuously. The analysis throws no light on the rates occurring at puberty because of lack of sufficient data and possibly because of inherent inadequacies in the method.

Whether or not the assumptions of the method suggested by Samuel Brody are valid for this type of problem, it is clear that the use of average relative growth rates is inadequate. In the future linear facial growth will have to be analyzed by means of infinitesimals and instantaneous growth rates.

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