The Dental Arch Form Revisited

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Abstract: Recently, the beta function has been shown to be an accurate mathematical model of the human dental arch. In this research, we tried to find the equation of a curve that would be similar to the generalized beta function curve and at the same time could represent tapered, ovoid, and square dental arches. A total of 23 sets of naturally well-aligned Class I casts were selected, and the depths and widths of the dental arches were measured at the canine and second molar regions. Using the mean depths and widths, functions in the form of $Y = AX^m + BX^n$ were calculated that would pass through the central incisors, canines, and second molars. Each function was compared with the generalized beta function with the use of root mean square values. It was shown that the polynomial function $Y = AX^6 + BX^2$ was the nearest to the generalized beta function. Then the coordinates of the midincisal edges and buccal cusp tips of each dental arch were measured, and the correlation coefficient of each dental arch with its corresponding sixth order polynomial function was calculated. The results showed that the function $Y = AX^6 + BX^2$ could be an accurate substitute for the beta function in less common forms of the human dental arch. (*Angle Orthod* 2001;71:386–389.)

Key Words: Arch form; Beta function; Equation; Formula; Polynomial; Correlation coefficient

INTRODUCTION

Many geometric forms and mathematical functions have been proposed as models of the human dental arch.[1–25] However, it has become clear that the models defined by 1 parameter cannot describe the dental arch form accurately.[26]

Recently, it has been shown that the human dental arch form is accurately represented mathematically by the beta function.[27] Two parameters, the depth and the width of the dental arch at the second molar region, define this model. When using the beta function model, if the measured value for width is increased 1 mm and the measured value of depth increased 1.5 mm, the resulting equation will be an excellent representation of the dental arch shape, including the second molars. In addition, it will be an excellent generalized equation of the maxillary and mandibular arch shapes for each of the Angle classifications of occlusions.[27]

We see naturally well-aligned human dental arches with different shapes. They are roughly categorized as square, ovoid, and tapered in prosthodontics. Although these arch forms have not yet been accurately defined, they do exist in nature.[28]

We tried to present a model that is defined by 4 parameters, ie, by the depths and widths of the dental arch at the canine and second molar regions. Such a model will be flexible at the anterior as well as posterior regions of a dental arch. Many mathematical models can be found that are defined by these 4 parameters, but we tried to find the one that is more compatible with the beta function and, consequently, with the dental arch. Therefore, this model, while representing the dental arch with high accuracy, will represent the square and tapered dental arches as well as ovoid ones.

MATERIALS AND METHODS

Twenty-three sets of pretreatment orthodontic models of fully developed Class I dentitions were selected. Casts exhibiting attrition, fracture, ectopic eruption, crowding, or midline deviations were not included. The depths and widths of the dental arches at the canine and second molar regions were measured using an electronic gauge with an accuracy of 0.1 mm. The depths and widths were defined as follows:

1. intersecond molar width (Wm): distance between the distobuccal cusp tips of the second molars,
2. intercanine width (Wc): distance between the canine cusp tips,
3. second molar depth (Dm): distance between the contact

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TABLE 1. Root Mean Square (RMS) Values of the Calculated Functions; for Odd Powers, the Absolute Value of x Was Used

Function	Maxillary RMS	Mandibular RMS	Mean
Y(2, 1)	1.5659	1.544	1.555
Y(3, 1)	0.8734	0.8982	0.8858
Y(3, 2)	1.0916	0.9706	1.0311
Y(4, 1)	1.0757	1.3965	1.2361
Y(4, 2)	0.9078	0.892	0.8999
Y(4, 3)	1.9713	1.5081	1.7397
Y(5, 1)	1.533	2.1333	1.8332
Y(5, 2)	0.7565	0.9378	0.8472
Y(5, 3)	2.4239	1.8778	2.1509
Y(5, 4)	4.4073	4.1418	4.2746
Y(6, 1)	2.007	2.8286	2.4178
Y(6, 2)	0.6058	1.072	0.8389
Y(6, 3)	2.8656	2.2649	2.5653
Y(6, 4)	5.6736	5.6612	5.6674
Y(6, 5)	9.0003	10.0614	9.5309

of the central incisors and a line that connects the distobuccal cusp tips of the second molars,

4. canine depth (Dc): distance between the contact of the central incisors and a line that connects the canine cusp tips.

Two operators measured each distance and, when there was a difference between the measurements of the 2 operators, the mean value was used. The mean depths and widths were calculated for the maxillary and mandibular casts, respectively. The values of *A* and *B* in the formula $Y(m, n) = AX^m + BX^n$ were calculated so that the curve of this formula would pass exactly through the contact of the central incisors, canine cusp tips, and the distobuccal cusp tips of the second molars. The values were calculated using the mean depths and widths of the dental arches.

Then the corresponding maxillary and mandibular generalized beta functions were calculated using the mean depths and widths of the dental arches. Using this method, different formulas of the form $Y(m, n) = AX^m + BX^n$ were calculated. All of the resulting formulas represented the curves that would exactly pass through the contact of the central incisors, canine cusp tips, and the distobuccal cusp tips of the second molars.

To find the formula that is most similar to the beta function—and consequently to the dental arch—we calculated the root mean square values (RMS). RMS is the standard mathematical tool by which the similarity of 2 curves is evaluated (the more the similarity of 2 curves, the lower the value of the RMS).

A computer program was written to calculate the values of *A, B,* and RMS for each of the $Y(m, n)$ functions. The calculations showed that $Y(6, 2)$ was the nearest to the beta function (Table 1). As the next step, the occlusal surfaces of all the casts were photocopied (Xerox 1050, Rank Xerox Ltd) with a 100-mm ruler in the field to allow for calculation of enlargement.

Measurement of the 100-mm ruler image in each pho-
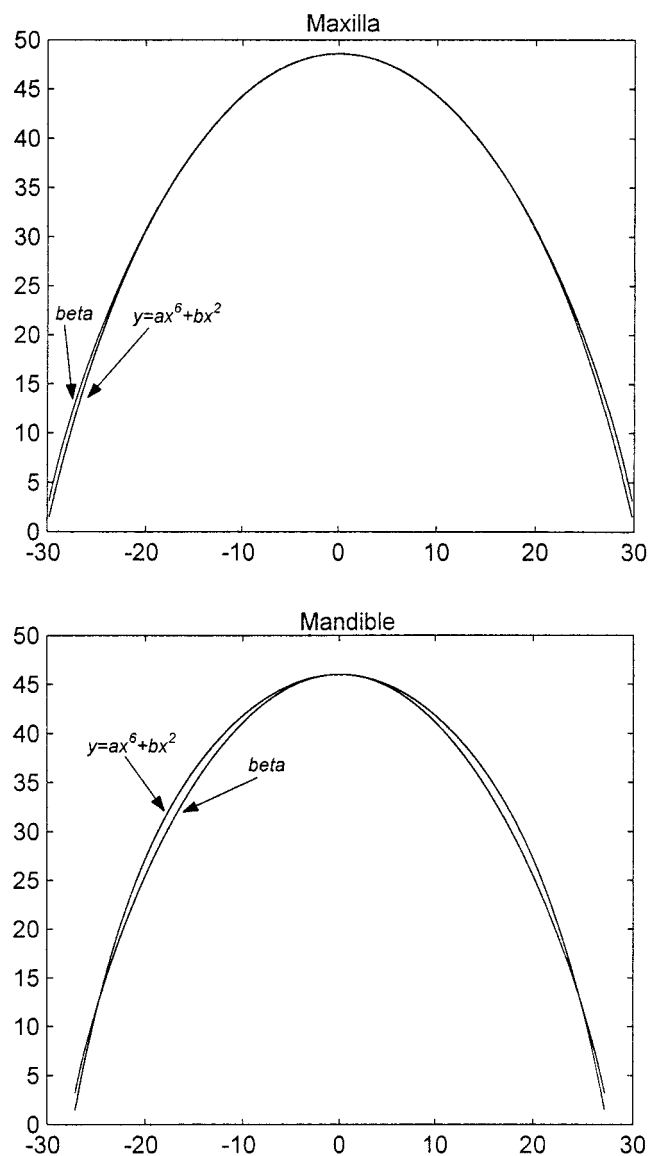
FIGURE 1. (A) Maxillary and (B) mandibular curves of the generalized beta function and the equation $y = ax^6 + bx^2$ for the mean measured data.

tocopy revealed that the copier resulted in an average enlargement of the image of 4%, which was not statistically significant.[29] The reliability of the use of photocopies of the occlusal surfaces of dental models has been demonstrated previously.[29] The photocopies were scanned using a flatbed transparency scanner (Genius Vivid III, Rank Xerox Ltd), and a digital image of the occlusal surface of each cast was prepared.

These digital images were transferred to AutoCAD environment (1982–1999 AutoDesk, Inc), and the coordinates of the midincisal edges and buccal cusp tips of all the teeth were determined. Then the polynomial function (in the form of $Y = AX^6 + BX^2$) corresponding to each dental arch was determined, and the correlation coefficient between each dental arch and its corresponding sixth order polyno-

TABLE 2. Correlation Coefficient Between the Function $Y = AX^6 + BX^2$ and Each Dental Arch

	Maxilla			Mandible		
Cast	A	B	Correlation Coefficient (r)	A	B	Correlation Coefficient (r)
1	5.63×10^{-8}	0.03263	0.99	1.17×10^{-7}	0.02922	0.97
2	2.81×10^{-9}	0.04767	0.97	1.38×10^{-8}	0.0475	0.96
3	-1.13×10^{-8}	0.05935	0.95	-1.36×10^{-8}	0.067	0.92
4	-6.93×10^{-9}	0.05568	0.98	-1.85×10^{-8}	0.06804	0.97
5	-2.57×10^{-9}	0.04516	0.98	-3.58×10^{-9}	0.05	0.98
6	1.08×10^{-9}	0.04802	0.97	1.14×10^{-8}	0.047	0.98
7	4.56×10^{-9}	0.04886	0.99	5.58×10^{-8}	0.03719	0.98
8	1.52×10^{-8}	0.03520	0.95	3.25×10^{-8}	0.03561	0.96
9	-2.12×10^{-10}	0.04947	0.99	1.83×10^{-9}	0.0551	0.97
10	1.85×10^{-8}	0.03930	0.97	3.93×10^{-8}	0.03971	0.97
11	5.14×10^{-8}	0.03947	0.98	1.81×10^{-7}	0.03470	0.99
12	4.73×10^{-9}	0.04558	0.98	1.08×10^{-8}	0.04810	0.98
13	2.22×10^{-8}	0.04131	0.97	7.03×10^{-8}	0.02965	0.97
14	-6.98×10^{-9}	0.05456	0.98	-4.11×10^{-9}	0.05451	0.98
15	4.02×10^{-8}	0.03316	0.99	9.98×10^{-8}	0.02774	0.97
16	-4.02×10^{-9}	0.05425	0.97	-6.33×10^{-10}	0.05887	0.95
17	1.56×10^{-8}	0.0444	0.98	2.65×10^{-8}	0.04958	0.99
18	4.99×10^{-8}	0.03732	0.98	1.34×10^{-7}	0.02798	0.98
19	4×10^{-9}	0.04284	099	1.84×10^{-8}	0.03923	0.98
20	2.55×10^{-8}	0.03675	0.97	6.54×10^{-8}	0.02933	0.95
21	7.59×10^{-8}	0.02197	0.98	1.67×10^{-7}	0.01188	0.99
22	-2.58×10^{-9}	0.0444	0.98	2.37×10^{-8}	0.03584	0.99
23	7.32×10^{-8}	0.0131	0.98	1.48×10^{-7}	8.55×10^{-3}	0.98
Mean			0.98			0.97
SD			0.01			0.02
Maximum			0.99			0.99
Minimum			0.95			0.92

mial function was calculated. The coordinates of 18 points on each dental arch (ie, midincisal edges and buccal cusp tips of the teeth) were used to calculate the correlation coefficient.

RESULTS

The values of RMS are shown in Table 1 and the correlation coefficients are presented in Table 2. It is seen that the RMS of $Y(6, 2)$ is the smallest value and $Y(5, 2)$ and $Y(4, 2)$ take the next positions (Figure 1). As is seen in Table 2, the mean correlation coefficient is 0.98 with a standard deviation of 0.02.

DISCUSSION

The reader might reasonably ask why this research did not deal with equations of greater degrees. One might assume that, if $y = ax^6 + bx^2$ is a good approximation, equations of greater degrees should be even better. In theory, this is quite true. Theoretically, there exists a unique polynomial equation of degree $n + 1$ or less (where n is the number of data points) that will fit the data exactly. Such curves accurately fit the data points, but they tend to be wavy rather than smooth. As n increases, the likelihood of obtaining a completely accurate curve fit without this wavelike property is reduced to nearly zero. Even if a smooth

curve of high degree could be obtained, measurement errors and round-off error make such computations unwieldy, if not altogether impossible.[21] In fact, we examined all polynomial functions (in the form of $y = ax^m + bx^n$) of degree less than or equal to 10, but for this reason and for the sake of brevity, they are not included in the results. Here, these 5 points (ie, the contact of the central incisors, canine cusp tips, and the distobuccal cusp tips of the second molars) are similar to the knots of a cubic spline function[23] and we chose the form of $y = ax^m + bx^n$ because it could be accurately adjusted to pass through these 5 points.

The high correlation coefficient shows that this function can be an accurate mathematical model of the dental arch. However, it should be noticed that this study has been performed on Class I cases and further studies are needed to determine the fitness of this function for Class II and Class III occlusions.

The square, ovoid, and tapered arch forms have not yet been mathematically defined. One solution may be to define them based on the relative ratios of the canine and the second molar cross-arch widths along with their relative arch depths.

When the Wc/Wm ratio increases or the Dc/Dm ratio decreases, the arch becomes squarer. On the contrary, when Wc/Wm ratio decreases or Dc/Dm ratio increases, the arch gets a more tapered form. Therefore, the (Wc/Wm) × (Dc/

$Dm)^{-1}$ ratio may be able to describe the arch form. When this ratio of a dental arch is within the range of mean $\pm$ 1 SD, we can assume the arch form is ovoid. However, when this ratio for an arch form is more than mean + 1 SD (eg, casts 18, 21, 23), we can consider the arch form as square. Finally, when the ratio is less than mean + 1 SD (eg, casts 4, 14), we can consider the arch form as tapered.

It has been shown that the beta function alone is insufficient to accurately describe an expanded dental arch approximating the square arch form. Consequently, the arch form is described by 2 equations, ie, the hyperbolic cosine function for the 6 anterior teeth and the beta function for the posterior teeth.[30] By the sixth power function presented here, such an arch form may be defined by only 1 equation.

CONCLUSION

By measuring the 4 parameters of a dental arch (ie, Dc, Wc, Dm, and Wm), one can simply calculate the $y = ax^6 + bx^2$ corresponding to the dental arch. The curve of this equation can be wide or narrow in the anterior as well as posterior regions of a dental arch and so can be an accurate substitute for the beta function in less common, ie, square or tapered, forms of the human dental arch.

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